Journal of Statistical Physics, Vol. 66, Nos. 1/2, 1992

Higher-Order Implicit Strong Numerical Schemes for Stochastic Differential Equations

P. E. Kloeden[1] **and E. Platen**[2]

Received June 12, 1990; final July 17, 1991

Higher-order implicit numerical methods which are suitable for stiff stochastic differential equations are proposed. These are based on a stochastic Taylor expansion and converge strongly to the corresponding solution of the stochastic differential equation as the time step size converges to zero. The regions of absolute stability of these implicit and related explicit methods are also examined.

KEY WORDS: Stiff stochastic differential equations; numerical simulations; strong order of convergence; implicit and fully implicit schemes; stochastic Taylor formula.

1. INTRODUCTION

Stochastic differential equations are being increasingly used to model the effects of noise on complex physical systems. Since analytic solutions are rarely available in practical situations, numerical simulations of the equations are required. However, heuristic adaptations of deterministic numerical methods to stochastic differential equations have been found to have serious shortcomings. Thus the development of efficient numerical methods for stochastic differential equations has become essential if significant advances are to be made in this field. Greiner *et al.*[6] have discussed the issues involved in applying numerical methods to stochastic differential equations and have indicated many questions which remain to be resolved. An extensive review of the existing literature on this subject was given in Kloeden and Platen.[10] Other surveys can be found in Pardoux and Talay[17] and Milstein.[16]

[1] Department of Computing and Mathematics, Deakin University, Geelong, Victoria 3217, Australia.

[2] Karl-Weierstrass-Institut für Mathematik, Mohrenstrasse 39, 1086 Berlin, Germany; and Australian National University, SMS, SRS, GPO Box 4, ACT 2605 Canberra, Australia.

0022-4715/92/0100-0283$06.50/0

The numerical stability of a scheme is often crucial for its successful application. It is well known from deterministic numerical analysis that the numerical integration of stiff systems requires the use of implicit methods which are much more stable numerically. Many important physical problems in a stochastic setting also involve stiff systems, for which several authors have applied implicit stochastic numerical schemes. In particular, we mention the papers of Klauder and Petersen,[9] Smith and Gardiner,[24] McNeil and Craig,[14] Drummond and Mortimer,[5] Hernandez and Spigler,[8] and Petersen.[18] Earlier results in this direction can be found in Talay[25] and Milstein.[16]

We shall begin this paper with a precise definition of stiffness for stochastic differential equations which reduces to the usual definition for a deterministic differential equation in the absence of noise. Then we systematically describe higher-order numerical schemes which are strongly convergent and based on the stochastic Taylor expansion. Such schemes are essential for the direct simulation of trajectories of a diffusion process X solving a stochastic differential equation such as in nonlinear filtering or the simulation of stochastic flows where good approximations of the paths are required. As in Kloeden and Platen,[10] we say that an approximation Y^δ of a solution X converges with strong order $\gamma > 0$ as the time step $\delta \to 0$ if there exists a finite constant K not depending on δ such that

$$E|X_T - Y_T^\delta| = \langle |X_T - Y_T^\delta| \rangle \leqslant K\delta^\gamma$$

Here the approximation Y^δ must always be generated by the same trajectory of the Wiener process as the sample path of the solution X of the stochastic differential equation. In Section 8 we shall state a theorem involving conditions for pathwise convergence. In contrast, to approximate functionals of the solution such as $Eg(X_T))$ we need only weak approximations which are often easier to use as the random variables simulating the noise can take a simpler form. As in Kloeden and Platen,[10] we say that an approximation Y^δ converges weakly with order $\beta > 0$ as $\delta \to 0$ if for each polynomial g there exists a finite constant K_g not depending on δ such that

$$|Eg(X_T) - Eg(Y_T^\delta)| = |\langle g(X_T) \rangle - \langle g(Y_T^\delta) \rangle| \leqslant K_g \delta^\beta$$

Here we are essentially approximating the probability measure induced by the solution X. We note that a weak scheme of a given order will usually differ in structure from a strong scheme of the corresponding order.

We shall present new families of implicit schemes which are well suited to stiff stochastic systems and avoid some of the pitfalls inherent in fully implicit schemes that have been suggested by earlier authors. We shall define and investigate numerical stability, asymptotic numerical stability, A-stability, and regions of absolute stability for these stochastic numerical schemes. This concept of A-stability and its results turn out to be direct transformations of their well-known deterministic counterparts to the stochastic context. Moreover, an A-stable stochastic scheme is also A-stable in the deterministic sense when applied to ordinary differential equations.

In addition we shall derive some two-step strong stochastic schemes. Finally, we shall present some numerical results, which clearly indicate the improved stability and convergence order of the implicit schemes.

2. STIFF STOCHASTIC DIFFERENTIAL EQUATIONS

We shall consider a d-dimensional Ito stochastic differential equation

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t \tag{2.1}$$

for $0 \leqslant t \leqslant T$ with initial value $X_0 \in \mathscr{R}^d$, where $a(x) = \{a^k(x)\}_{k=1}^d$ is the d-dimensional drift and $b(x) = \{b^k(x)\}_{k=1}^d$ is the d-dimensional diffusion coefficient. Here $W = \{W_t, t \geqslant 0\}$ denotes a standard scalar Wiener process with

$$EW_t = \langle W_t \rangle = 0$$

and

$$EW_s W_t = \langle W_s W_t \rangle = \min\{s, t\}$$

for $s, t \geqslant 0$. Most of the results of this paper also hold true for a multi-dimensional Wiener process and for time-dependent drift and diffusion coefficients, but we shall restrict attention to the above case in order to simplify the exposition.

From the mathematical viewpoint it is more convenient to write (2.1) in integral form as an Ito stochastic equation

$$X_t = X_0 + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s \tag{2.2}$$

where the second integral is an Ito integral, which does not follow the rules of ordinary calculus. Correcting the drift term to

$$\underline{a}(x) = a(x) - \frac{1}{2} \sum_{k=1}^{d} b^k(x) \frac{\partial b}{\partial x^k}(x)$$

we can also express this with respect to a Stratonovich integral, which we shall distinguish with a $\circ$, obtaining the corresponding Stratonovich stochastic equation

$$X_t = X_0 + \int_0^t \underline{a}(X_s)\, ds + \int_0^t b(X_s) \circ dW_s \tag{2.3}$$

for the same Ito process $X = \{X_t, 0 \leqslant t \leqslant T\}$ as in (2.2). For further details on stochastic calculus and stochastic differential equations we refer to Arnold[1] or Kloeden and Platen.[12]

We shall assume that the drift and diffusion coefficients a and b satisfy Lipschitz and linear growth conditions, to ensure the existence and uniqueness of a strong solution of (2.2). For the proofs of convergence of higher-order numerical schemes we shall also suppose that a and b are sufficiently smooth and that all moments of the initial value X_0 exist, that is,

$$E|X_0|^q = \langle |X_0|^q \rangle < \infty \tag{2.4}$$

for $q = 1, 2, \ldots$.

In order to define stiff stochastic differential equations, we need to introduce Lyapunov exponents. We suppose that the Stratonovich equation (2.3) has a stochastic stationary solution $\bar{X}_t$ (see Hasminski[7]) and then linearize about $\bar{X}_t$ to obtain the linearized system

$$Z_t = Z_0 + \int_0^t \underline{A}(s) Z_s\, ds + \int_0^t B(s) Z_s \circ dW_s \tag{2.5}$$

where $\underline{A}$ and B are random $d \times d$ matrices defined componentwise by

$$\underline{A}(t)^{ij} = \frac{\partial \underline{a}^i}{\partial x^j} (\bar{X}_t) \tag{2.6}$$

and

$$B(t)^{ij} = \frac{\partial b^i}{\partial x^j} (\bar{X}_t) \tag{2.7}$$

for $i, j = 1, \ldots, d$. Then by the multiplicative ergodic theorem of Osceledec (see Arnold and Wihstutz[2]) there exist d nonrandom Lyapunov exponents

$$\lambda_d \leqslant \lambda_{d-1} \leqslant \cdots \leqslant \lambda_1$$

and a partitioning of $\mathscr{R}^d$ into random subsets $E_d(\omega)$, $E_{d-1}(\omega), \ldots, E_1(\omega)$ such that for the solutions of (2.5) starting in these sets the limits

$$\lambda(Z_0) = \lim_{t \to \infty} \frac{1}{t} \ln |Z_t| \tag{2.8}$$

take the values $\lambda_d, \lambda_{d-1}, \ldots, \lambda_1$, respectively.

We shall say that the linear stochastic equation (2.5) is *stiff* if its Lyapunov exponents satisfy

$$\lambda_d \ll \lambda_1 \tag{2.9}$$

More generally, we shall say that a stochastic equation (2.3) is *stiff* if its linearization (2.5) about some stationary solution is stiff. Note that if the original equation (2.3) has additive noise on linearizing about a stationary solution, we obtain a degenerate linear equation (2.5) in which $B(t) \equiv 0$ but $\underline{A}(t)$ is a random matrix-valued function.

This generalizes the deterministic concept of stiffness, since the real parts of the eigenvalues of the coefficient matrix of a deterministic linear differential equation are its Lyapunov exponents. Moreover, a stiff deterministic equation is also stiff in the stochastic sense. Thus stochastic stiffness also refers to the behavior of solutions having two or more widely different time scales. It is this large difference in time scales that gives rise to difficulties in numerical investigations of stiff deterministic systems as well as stiff stochastic systems.

3. STRONG CONVERGENCE AND STABILITY OF STOCHASTIC NUMERICAL SCHEMES

For convenience we shall restrict our attention to equidistant time discretizations of the interval $[0, T]$ with points

$$\tau_n = n\Delta \tag{3.1}$$

for $n = 1, \ldots, n_T$ with step size $\Delta = T/n_T$ for some $n_T = 1, 2, \ldots$. The simplest time discrete approximation then is given by the Euler scheme

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n \tag{3.2}$$

for $n = 0, 1, 2, \ldots$, with $Y_0^\Delta = X_0$, where $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ is the increment in the Wiener process and is normally distributed with zero mean and variance $E(\Delta W_n)^2 = \langle (\Delta W_n)^2 \rangle = \Delta$. This scheme provides a recursive algorithm for simulating approximate solutions of the stochastic equation (2.2) and hence also (2.3).

In this paper we shall only consider approximations which converge strongly to the solutions of (2.2) and (2.3), that is, with the trajectories becoming closer and closer as the step sizes Δ decreases to 0. To be precise, we shall say that a time discrete approximation Y^Δ converges strongly with the strong order $\gamma > 0$ to the corresponding solution X of the stochastic equation (2.2) if the estimate

$$E|X_T - Y_{n_T}^\Delta| = \langle |X_T - Y_{n_T}^\Delta| \rangle \leqslant K\Delta^\gamma \tag{3.3}$$

holds for each $\Delta \in (0, 1)$, where K is a positive constant which does not depend on Δ.

The criterion (3.3) is thus a straightforward generalization of the order of convergence of deterministic schemes, to which it reduces in the absence of noise. However, it turns out under Lipschitz and linear growth conditions on a and b that the Euler scheme (3.2) converges with the strong order $\gamma = 0.5$, in contrast with the order 1.0 for the deterministic Euler scheme (e.g., see Platen[19]). This is a direct consequence of the differences in the deterministic and stochastic calculi.

To handle stiff stochastic differential equations in particular and error propagation in general we need a counterpart of the deterministic concept of numerical stability for stochastic numerical schemes.

Let Y^Δ denote a time discrete approximation with step size $\Delta \in (0, 1)$ which starts at Y_0^Δ at time $t = 0$ and let $\bar{Y}^\Delta$ be the corresponding approximation which starts at $\bar{Y}_0^\Delta$. We shall say that the time discrete approximation Y^Δ is *stochastically numerically stable* for a given stochastic equation (2.2) if for any time interval $[0, T]$ there exists a positive constant Δ_0 such that

$$\lim_{|Y_0^\Delta - \bar{Y}_0^\Delta| \to 0} \max_{0 \leqslant n \leqslant n_T} P(|Y_n^\Delta - \bar{Y}_n^\Delta| \geqslant \varepsilon) = 0 \tag{3.4}$$

for each $\varepsilon > 0$ and each $\Delta \in (0, \Delta_0)$, where P denotes the underlying probability measure. Essentially, this says that the scheme is continuous in initial conditions uniformly on finite time intervals. More generally, we shall say that the time discrete approximation is *stochastically numerically stable* if it is stochastically numerically stable for the class of stochastic equations for which the approximate solution converges strongly with some order $\gamma > 0$ to the corresponding solution of the stochastic equation. All of the one-step schemes proposed in this paper will turn out to be stochastically numerically stable.

We emphasize that the stochastic numerical stability criterion applies only to step sizes $\Delta > 0$ that are less than some critical value Δ_0 which usually depends on both the particular time interval $[0, T]$ and the stochastic equation under consideration. However, this critical value may be extremely small in some cases. As the time interval $[0, T]$ becomes relatively large, the propagated error of what is nominally a stochastic numerically stable scheme may in fact become so unrealistically large as to make the approximation useless for some practical purpose. To exclude such cases, we shall say that a time discrete approximation Y^Δ is *asymptotically numerically stable* for a given stochastic equation if there exists a positive constant Δ_a such that

$$\lim_{|Y_0^\Delta - \bar{Y}_0^\Delta| \to 0} \lim_{T \to \infty} P(\max_{0 \leqslant n \leqslant n_T} |Y_n^\Delta - \bar{Y}_n^\Delta| \geqslant \varepsilon) = 0 \tag{3.5}$$

for each $\varepsilon > 0$ and each $\Delta \in (0, \Delta_a)$, where we have used the same notation as in (3.4) (recall that $T/n_T = \Delta$ here). In general we can only have asymptotic numerical stability when the stochastic differential equation has an asymptotically stable, stationary solution such as an ergodic solution.

As with the A-stability of deterministic differential equations,[4] we can also consider asymptotic numerical stability of a stochastic scheme with respect to an appropriately restricted class of stochastic differential equations. We shall choose the class of complex-valued linear test equations

$$dX_t = \lambda X_t \, dt + dW_t \tag{3.6}$$

where the parameter λ is a complex number with real part $\mathrm{Re}(\lambda) < 0$ and W is a real-valued standard Wiener process. This represents a simple stochastic generalization by including additive noise in the deterministic test equations used to test for the A-stability of deterministic schemes. Also in the stochastic case the critical value Δ_a will depend on the parameter λ. We know from Hasminski[7] that (3.6) has an ergodic solution when $\mathrm{Re}(\lambda) < 0$, which makes these equations a good choice of test equations for situations involving additive noise as well as for other noises.

We can write (3.6) as a 2-dimensional Ito stochastic differential equation with linear drift and constant diffusion coefficients in terms of the components (X_1, X_2), where $X = X_1 + iX_2$, namely

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW_t$$

where $\lambda = \lambda_1 + i\lambda_2$.

Suppose that we can write a given scheme with equidistant step size Δ applied to the test equations (3.6) with $\mathrm{Re}(\lambda) < 0$ in the recursive form

$$Y_{n+1}^{\Delta} = G(\lambda\Delta)\, Y_n^{\Delta} + Z_n^{\Delta} \tag{3.7}$$

for $n = 0, 1, \ldots$, where G is a mapping of a subset of the complex numbers into itself and the $Z_0^{\Delta}, Z_1^{\Delta}, \ldots$ are random variables which do not depend on λ or the $Y_0^{\Delta}, \ldots, Y_{n+1}^{\Delta}$. Then we shall call the set of complex numbers $\lambda\Delta$ with

$$\lambda_1 = \mathrm{Re}(\lambda) < 0 \qquad \text{and} \qquad |G(\lambda\Delta)| < 1 \tag{3.8}$$

the *region of absolute stability* of the scheme. For example, the region of absolute stability of the Euler scheme (3.2) is the interior of the unit circle with center at $-1 + 0i$, which is the same as for the deterministic Euler scheme. From the region of absolute stability of a scheme we can determine the appropriate equidistant step size Δ such that an error in the approximation by this scheme of a particular test equation from the class (3.6) will not grow in subsequent iterations. Obviously, a scheme is asymptotically numerically stable for such a test equation if $\lambda\Delta$ belongs to its region of absolute stability.

Generalizing the deterministic definition, we shall say that a stochastic scheme is *A-stable* if its region of absolute stability is the whole left half of the complex plane, that is, if it consists of all $\lambda\Delta$ with $\mathrm{Re}(\lambda)<0$ and $\Delta>0$. Thus the *implicit Euler scheme*

$$Y_{n+1}^{\Delta}=Y_n^{\Delta}+a(Y_{n+1}^{\Delta})\Delta+b(Y_n^{\Delta})\Delta W_n \tag{3.9}$$

is A-stable, whereas the Euler scheme (3.2) is not. Clearly, an A-stable stochastic scheme will be A-stable in the deterministic sense for an ordinary differential equation.[4]

The test equation (3.6) can be interpreted as a linearization of a damped oscillator with noise added. In this sense the concept of A-stability is related to additive noise. Various problems arise, however, when we consider multiplicative noise. For instance, if we apply the *fully implicit Euler scheme*

$$Y_{n+1}^{\Delta}=Y_n^{\Delta}+a(Y_{n+1}^{\Delta})\Delta+b(Y_{n+1}^{\Delta})\Delta W_n \tag{3.10}$$

to the 1-dimensional homogeneous linear Ito stochastic differential equation

$$dX_t=aX_t\,dt+bX_t\,dW_t$$

we obtain

$$Y_n^{\Delta}=Y_0^{\Delta}\prod_{k=0}^{n-1}\frac{1}{1-a\Delta-b\Delta W_k}$$

This expression is, however, not suitable as an approximation because one of its factors may become infinite. In fact, the first absolute moment $E(|Y_n^{\Delta}|)$ does not exist. It seems then that fully implicit schemes such as (3.10) will be not practicable, except perhaps in special cases such as for a linear equation with a strongly attracting drift and very weak noise intensity. In this paper we shall thus concentrate on implicit strong approximations which are implicit only in those terms containing nonrandom variables such as Δ or Δ^2.

4. STOCHASTIC TAYLOR SCHEMES

Truncated stochastic Taylor expansions provide a general systematic means of deriving numerical schemes for stochastic differential equations.

These are based on the Ito–Taylor formula proposed by Wagner and Platen[19,20,26] or on the Stratonovich–Taylor formula presented by Kloeden and Platen.[10,12] In both of these formula functions of an Ito process are represented in terms of multiple stochastic integrals. It is also possible to derive expansions containing implicit terms, which are required for implicit schemes, by repeated application of these stochastic Taylor formulas. We shall describe this procedure in the final section of the paper, where we shall also indicate the proof of strong convergence of the proposed schemes.

The simplest useful scheme that can be derived in this way is the Euler scheme (3.2). If we interpolate between it and the implicit Euler scheme (3.9), we obtain a *family of implicit Euler schemes*

$$Y_{n+1} = Y_n + [\alpha a(Y_{n+1}) + (1-\alpha)\, a(Y_n)]\Delta + b(Y_n)\,\Delta W_n \tag{4.1}$$

where $\alpha \in [0, 1]$ is the degree of implicitness. We note that for $\alpha = 0$ we have the Euler scheme (3.2); for $\alpha = 1$ the implicit Euler scheme (3.9); and for $\alpha = 1/2$ the scheme (4.1) generalizes the well-known deterministic trapezoidal method. Under Lipschitz and linear growth conditions on a and b the scheme (4.1) has the strong order $\gamma = 0.5$. Moreover, α can be chosen differently for each component if desired.

An order-1.0 strong Taylor scheme is the *Milstein scheme.*[15,16] In the 1-dimensional case it has the *Ito version*

$$Y_{n+1} = Y_n + a\Delta + b\,\Delta W_n + \tfrac{1}{2} bb'[(\Delta W_n)^2 - \Delta] \tag{4.2}$$

and the *Stratonovich version*

$$Y_{n+1} = Y_n + \underline{a}\Delta + b\,\Delta W_n + \tfrac{1}{2} bb'(\Delta W_n)^2 \tag{4.3}$$

where $\underline{a}$ is the corrected drift from the Stratonovich equation (2.3). Here we have abbreviated $f(Y_n)$ as f for any function f, which we shall continue to do in the rest of the paper.

We mentioned in Section 3 that it generally only makes sense to construct schemes with the implicit terms occurring in those terms involving nonrandom variables such as Δ, but not, for instance, ΔW_n. With this in mind we have a 1-dimensional *implicit Milstein scheme*, which has the *Ito version*

$$Y_{n+1} = Y_n + a(Y_{n+1})\Delta + b\,\Delta W_n + \tfrac{1}{2} bb'[(\Delta W_n)^2 - \Delta] \tag{4.4}$$

and the *Stratonovich version*

$$Y_{n+1} = Y_n + \underline{a}(Y_{n+1})\Delta + b\,\Delta W_n + \tfrac{1}{2} bb'(\Delta W_n)^2 \tag{4.5}$$

By interpolation in the general d-dimensinal case we obtain a *family of implicit Milstein schemes* with *Ito version*

$$Y_{n+1} = Y_n + [\alpha a(Y_{n+1}) + (1-\alpha)a]\Delta + b\,\Delta W_n + \tfrac{1}{2}L^1 b[(\Delta W_n)^2 - \Delta] \tag{4.6}$$

and *Stratonovich version*

$$Y_{n+1} = Y_n + [\alpha \underline{a}(Y_{n+1}) + (1-\alpha)\underline{a}]\Delta + b\,\Delta W_n + \tfrac{1}{2}L^1 b(\Delta W_n)^2 \tag{4.7}$$

where we have used the differential operator

$$L^1 = \sum_{k=1}^{d} b^k \frac{\partial}{\partial x^k} \tag{4.8}$$

All of these Milstein and implicit Milstein schemes (4.2)–(4.7) have strong order $\gamma = 1.0$ under sufficient regularity of the coefficients. These are Lipschitz and linear gowth conditions on $\underline{a}$, $(\partial/\partial x^k)\underline{a}$, b, $(\partial/\partial x^k)b$, and $(\partial/\partial x^k)^2 b$.

By including further terms from the Ito–Taylor expansion, we can achieve a strong order $\gamma = 1.5$ with the following scheme. For the 1-dimensional case the *order*-1.5 *strong Taylor scheme* is given by

$$\begin{aligned} Y_{n+1} = Y_n &+ a\Delta + b\,\Delta W_n + \tfrac{1}{2}bb'[(\Delta W_n)^2 - \Delta] \\ &+ ba'\,\Delta Z_n + \tfrac{1}{2}(aa' + \tfrac{1}{2}b^2 a'')\Delta^2 \\ &(ab' + \tfrac{1}{2}b^2 b'')(\Delta W_n\,\Delta - \Delta Z_n) \\ &+ \tfrac{1}{2}b(bb')'\,[\tfrac{1}{3}(\Delta W_n)^2 - \Delta]\,\Delta W_n \end{aligned} \tag{4.9}$$

Here an additional random variable ΔZ_n is required to represent the double stochastic integral

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2 \tag{4.10}$$

which is Gaussian-distributed with mean $E\,\Delta Z_n = \langle \Delta Z_n \rangle = 0$, variance $E(\Delta Z_n)^2 = \langle (\Delta Z_n)^2 \rangle = \frac{1}{3}\Delta^3$, and with correlation $E\,\Delta Z_n\,\Delta W_n = \langle \Delta Z_n\,\Delta W_n \rangle = \frac{1}{2}\Delta^2$. We remark that there is no difficulty in generating the pair $(\Delta W_n, \Delta Z_n)$ of correlated Gaussian random variables using the transformation

$$\Delta W_n = \zeta_{n,1}\sqrt{\Delta}, \qquad \Delta Z_n = \frac{1}{2}\left(\zeta_{n,1} + \frac{1}{\sqrt{3}}\zeta_{n,2}\right)\Delta^{3/2}$$

where $\zeta_{n,1}$ and $\zeta_{n,2}$ are independent standard $N(0, 1)$ distributed Gaussian random variables.

The corresponding *implicit order*-1.5 *strong Taylor scheme* in the 1-dimensional case takes the form

$$\begin{aligned}Y_{n+1} = {} & Y_n + a(Y_{n+1})\Delta \\ & - \tfrac{1}{2}[a(Y_{n+1})\, a'(Y_{n+1}) + \tfrac{1}{2}b(Y_{n+1})^2\, a''(Y_{n+1})]\Delta^2 \\ & + b\,\Delta W_n + \tfrac{1}{2}bb'[(\Delta W_n)^2 - \Delta] \\ & + (ab' + \tfrac{1}{2}b^2 b'' - ba')(\Delta W_n\, \Delta - \Delta Z_n) \\ & + \tfrac{1}{2}b(bb')'\, [\tfrac{1}{3}(\Delta W_n)^2 - \Delta]\, \Delta W_n \end{aligned} \tag{4.11}$$

Here we have made both the Δ and the Δ^2 terms in (4.9) implicit.

In the general d-dimensional case weobtain the *family of implicit order*-1.5 *strong Taylor schemes*

$$\begin{aligned}Y_{n+1} = {} & Y_n + [\alpha a(Y_{n+1}) + (1-\alpha)a]\Delta \\ & + (\tfrac{1}{2} - \alpha)[\beta L^0 a(Y_{n+1}) + (1-\beta)L^0 a]\Delta^2 \\ & + b\,\Delta W_n + L^1 a(\Delta Z_n - \alpha\,\Delta W_n\,\Delta) \\ & + L^0 b(\Delta W_n\,\Delta - \Delta Z_n) + \tfrac{1}{2}L^1 b[(\Delta W_n)^2 - \Delta] \\ & + \tfrac{1}{2}L^1 L^1 b[\tfrac{1}{3}(\Delta W_n)^2 - \Delta]\,\Delta W_n \end{aligned} \tag{4.12}$$

where $\alpha, \beta \in [0, 1]$ and, in addition to the previously defined L^1, we have used the differential operator

$$L^0 = \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} b^k b^l \frac{\partial^2}{\partial x^k\, \partial x^l}$$

The parameters α and β here can also take different values in each component of the vector equation (4.12). When $\alpha = 1/2$ the Δ^2 term vanishes and thus we obtain a relatively simple stochastic generalization of strong order 1.5 of the deterministic trapezoidal method. If we have additive noise, that is, with $b \equiv \text{const}$, then the last three terms vanish and (4.12) simplifies considerably. Finally, we note that a Stratonovich version of (4.12) is of little advantage, since it already involves most of the terms appearing in the following order-2.0 strong Taylor scheme.

The *order*-2.0 *strong Taylor scheme* in the 1-dimensional case has the form

$$Y_{n+1} = Y_n + \underline{a}\Delta + b\,\Delta W_n + \frac{1}{2}bb'(\Delta W_n)^2$$
$$+ b\underline{a}'\,\Delta Z_n + \frac{1}{2}\underline{a}\underline{a}'\,\Delta^2 + \underline{a}b'(\Delta W_n\,\Delta - \Delta Z_n)$$
$$+ \frac{1}{3!}\,b(bb')'\,(\Delta W_n)^3 + \frac{1}{4!}\,b(b(bb')')'\,(\Delta W_n)^4$$
$$+ \underline{a}(bb')'\,J_{(0,1,1),n} + b(\underline{a}b')'\,J_{(1,0,1),n}$$
$$+ b(\underline{a}'b')'\,J_{(1,1,0),n} \tag{4.13}$$

with the multiple Stratonovich integrals

$$J_{(0,1,1),n} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} ds_1 \circ dW_{s_2} \circ dW_{s_3} \tag{4.14}$$

$$J_{(1,0,1),n} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ\, dW_{s_1}\, ds_3 \circ dW_{s_3} \tag{4.15}$$

and

$$J_{(1,1,0),n} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ\, dW_{s_1} \circ dW_{s_2}\, ds_3 \tag{4.16}$$

The integrals (4.14)–(4.16), which are random variables, can be approximated as accurately as needed by a method described in Kloeden *et al.*[13] (see also Kloeden and Platen[12]), which uses series expansions of the Wiener process.

The 1-dimensional form of the *implicit order*-2.0 *strong Taylor scheme* is given by

$$Y_{n+1} = Y_n + \underline{a}(Y_{n+1})\Delta - \frac{1}{2}\underline{a}(Y_{n+1})\,\underline{a}'(Y_{n+1})\Delta^2$$
$$+ b\,\Delta W_n + \frac{1}{2}bb'(\Delta W_n)^2$$
$$+ (\underline{a}b' - b\underline{a}')(\Delta W_n\,\Delta - \Delta Z_n)$$
$$+ \frac{1}{3!}\,b(bb')'\,(\Delta W_n)^3 + \frac{1}{4!}\,b(b(bb')')'\,(\Delta W_n)^4$$
$$+ \underline{a}(bb')'\,J_{(0,1,1),n} + b(\underline{a}b')'\,J_{(1,0,1),n}$$
$$+ b(\underline{a}'b)'\,[J_{(1,1,0),n} - \tfrac{1}{2}(\Delta W_n)^2\Delta]$$

Finally, we have a *family of implicit order*-2.0 *strong Taylor schemes* which in the general d-dimensional case have the form

$$
\begin{aligned}
Y_{n+1} = Y_n &+ [\alpha \underline{a}(Y_{n+1}) + (1-\alpha)\underline{a}]\Delta \\
&+ \left(\frac{1}{2}-\alpha\right)[\beta \underline{L}^0 \underline{a}(Y_{n+1}) + (1-\beta)\underline{L}^0 \underline{a}]\Delta^2 \\
&+ b\,\Delta W_n + L^1 \underline{a}(\Delta Z_n - \alpha\,\Delta W_n\,\Delta) \\
&+ \underline{L}^0 b(\Delta W_n\,\Delta - \Delta Z_n) + \frac{1}{2} L^1 b(\Delta W_n)^2 \\
&+ \frac{1}{3!} L^1 L^1 b(\Delta W_n)^3 + \frac{1}{4!} L^1 L^1 L^1 b(\Delta W_n)^4 \\
&+ \underline{L}^0 L^1 b J_{(0,1,1),n} + L^1 \underline{L}^0 b J_{(1,0,1),n} \\
&+ L^1 L^1 \underline{a}[J_{(1,1,0),n} - \tfrac{1}{2}\alpha(\Delta W_n)^2\,\Delta]
\end{aligned}
\tag{4.17}
$$

where α, $\beta \in [0, 1]$ and

$$\underline{L}^0 = \sum_{k=1}^{d} \underline{a}^k \frac{\partial}{\partial x^k}$$

We note that for $\alpha = 1/2$ the third term in (4.18) vanishes and for additive noise the 6th to the 11th terms are no longer needed.

5. STOCHASTIC RUNGE-KUTTA-TYPE SCHEMES

In this section we shall consider implicit schemes which avoid the use of derivatives in the terms involving nondeterministic stochastic integrals. They are obtained from the corresponding implicit strong Taylor schemes by replacing the derivatives there by finite differences expressed in terms of appropriate supporting values. For this reason we shall call them implicit strong Runge-Kutta schemes, but it must be emphasized that they are not simply heuristic stochastic adaptations of the deterministic Runge-Kutta schemes.(8,23) Some preliminary explicit weak schemes of this kind for stochastic differential equations with constant diffusion coefficients can be found in Chang.(3)

For the 1-dimensional case an *implicit order*-1.0 *strong Runge-Kutta scheme* is

$$
\begin{aligned}
Y_{n+1} = Y_n &+ a(Y_{n+1})\Delta + b\,\Delta W_n \\
&+ \frac{1}{2\sqrt{\Delta}}[b(\bar{Y}_n) - b][(\Delta W_n)^2 - \Delta]
\end{aligned}
\tag{5.1}
$$

with supporting value

$$\bar{Y}_n = Y_n + a\Delta + b\sqrt{\Delta}$$

By interpolating between this scheme and the corresponding explicit scheme we can form a *family of implicit order*-1.0 *strong Runge-Kutta* schemes. In the d-dimensional case these have the form

$$Y_{n+1} = Y_n + [\alpha a(Y_{n+1}) + (1-\alpha)a]\Delta + b\,\Delta W_n + \frac{1}{2\sqrt{\Delta}}[b(\bar{Y}_n) - b][(\Delta W_n)^2 - \Delta] \tag{5.2}$$

with vector supporting value

$$\bar{Y}_n = Y_n + a\Delta + b\sqrt{\Delta}$$

and parameter $\alpha \in [0, 1]$.

There is also a *Stratonovich version*

$$Y_{n+1} = Y_n + [\alpha \underline{a}(Y_{n+1}) + (1-\alpha)\underline{a}]\Delta + \tfrac{1}{2}[b(\bar{\Psi}_n) + b]\,\Delta W_n \tag{5.3}$$

with vector supporting value

$$\bar{\Psi}_n = Y_n + \underline{a}\Delta + b\,\Delta W_n$$

and parameter $\alpha \in [0, 1]$. We remark that we still obtain convergence with the strong order $\gamma = 1.0$ if we omit the $\underline{a}\Delta$ term in the supporting value $\bar{\Psi}_n$.

In the d-dimensional case an *implicit order*-1.5 *strong Runge-Kutta scheme* is given by

$$\begin{aligned} Y_{n+1} = Y_n &+ \frac{1}{2}[a(Y_{n+1}) + a]\Delta + b\,\Delta W_n \\ &+ \frac{1}{2\sqrt{\Delta}}[a(\bar{Y}_+) - a(\bar{Y}_-)]\left(\Delta Z_n - \frac{1}{2}\Delta W_n \Delta\right) \\ &+ \frac{1}{4\sqrt{\Delta}}[b(\bar{Y}_+) - b(\bar{Y}_-)][(\Delta W_n)^2 - \Delta] \\ &+ \frac{1}{2\Delta}[b(\bar{Y}_+) - 2b + b(\bar{Y}_-)](\Delta W_n \Delta - \Delta Z_n) \\ &+ \frac{1}{4\Delta}\{b(\bar{\Phi}_+) - b(\bar{\Phi}_-) - [b(\bar{Y}_+) - b(\bar{Y}_-)]\} \\ &\times \left[\frac{1}{3}(\Delta W_n)^2 - \Delta\right]\Delta W_n \end{aligned} \tag{5.4}$$

with supporting values

$$\bar{Y}_{\pm} = Y_n + a\Delta \pm b\sqrt{\Delta}$$

and

$$\bar{\Phi}_{\pm} = \bar{Y}_{+} \pm b(\bar{Y}_{+})\sqrt{\Delta}$$

Here we have chosen the degree of implicitness $\alpha = 1/2$, which simplifies the scheme. We note that for additive noise only the first four terms remain in this scheme.

We shall only consider the additive noise case here for the strong order $\gamma = 2.0$ scheme. Then in the d-dimensional case we have the *implicit order-2.0 strong Runge-Kutta scheme for additive noise*

$$Y_{n+1} = Y_n + b\,\Delta W_n + \{\underline{a}(\bar{Y}_{+}) + \underline{a}(\bar{Y}_{-}) - \tfrac{1}{2}[\underline{a}(Y_{n+1}) + \underline{a}]\}\Delta \quad (5.5)$$

with

$$\bar{Y}_{\pm} = Y_n + \frac{1}{2}\underline{a}\Delta + \frac{1}{\Delta}b(\bar{\eta} \pm \bar{\zeta})$$

where

$$\bar{\eta} = \tfrac{1}{2}\Delta Z_n + \tfrac{1}{4}\Delta W_n\,\Delta$$

and

$$\bar{\zeta} = \left\{ J_{(1,1,0),n}\Delta - \frac{1}{2}(\Delta Z_n)^2 + \frac{1}{8}\left[(\Delta W_n)^2 + \frac{1}{2}\left(\frac{2\Delta Z_n}{\Delta} - \Delta W_n\right)^2\right]\Delta^2 \right\}^{1/2}$$

This scheme has a surprisingly simple structure in spite of its strong order $\gamma = 2.0$.

6. A-STABILITY OF STRONG SCHEMES

We defined the A-stability of a strong scheme in terms of the mapping G in Eq. (3.7). It is easy to determine this mapping for the lower-order schemes. In particular, for the family of implicit Euler and Milstein schemes (3.9) and (4.6), respectively, it is defined by

$$G(\lambda\Delta) = (1 - \alpha\lambda\Delta)^{-1}[1 + (1-\alpha)\lambda\Delta] \quad (6.1)$$

where $\alpha \in [0, 1]$ is the common degree of implicitness of all components. The inequalities (3.8) for the region of absolute stability are thus equivalent to

$$(1 - 2\alpha)(\lambda_1^2 + \lambda_2^2)\Delta^2 + 2\lambda_1\Delta < 0$$

where $\lambda_1 + i\lambda_2$. For $0 \leqslant \alpha < 1/2$ we can rewrite this as

$$(\lambda_1 \Delta + A)^2 + (\lambda_2 \Delta)^2 < A^2$$

where $A = (1 - 2\alpha)^{-1}$. Thus, the region of absolute stability is the interior of the circle of radius A which is centered at $-A + i0$, when $0 \leqslant \alpha < 1/2$. On the other hand, for $1/2 \leqslant \alpha < 1$ it is the whole left-hand side of the complex plane, so these schemes are then A-stable.

The implicit order-1.5 Runge-Kutta scheme (5.4) has the same G function (6.1) with $\alpha = 1/2$, so is also A-stable.

For the implicit order-2.0 Runge-Kutta scheme (5.5) we find that

$$G(\lambda\Delta) = (1 + \tfrac{1}{2}\lambda\Delta)^{-1}\,(1 + \tfrac{3}{2}\lambda\Delta + \lambda^2\Delta^2)$$

The corresponding region of absolute stability satisfies the polar coordinate inequality

$$4r \cos^2\theta + (2 + 3r^2)\cos\theta + r^3 < 0$$

with $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$, where $r = (\lambda_1^2 + \lambda_2^2)^{1/2}\Delta$ and $\theta = \arctan(\lambda_2/\lambda_1)$, so this scheme is not A-stable.

The order-1.5 and -2.0 strong Taylor schemes involve the term $L^0 a$, which is more easily determined using the real 2-dimensional form of the test equation (3.6) and turns out to equal $\lambda^2 X$ in complex notation. For these schemes

$$\begin{aligned} G(\lambda\Delta) = {} & [1 - \alpha\lambda\Delta - (\tfrac{1}{2} - \alpha)\beta\lambda^2\Delta^2]^{-1} \\ & \times [1 + (1-\alpha)\lambda\Delta + (\tfrac{1}{2} - \alpha)(1-\beta)\lambda^2\Delta^2] \end{aligned}$$

and that the region of absolute stability is given by the polar coordinate inequality

$$[2 + (1-\alpha)(1-2\alpha)r^2]\cos\theta + 2(1-2\alpha)r\cos^2\theta + \tfrac{1}{4}(1-2\alpha)^2(1-2\beta)r^3 < 0$$

with $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$. Hence the schemes are A-stable if and only if $\alpha = 1/2$, $\beta \in [0, 1]$, or $\alpha, \beta \in [1/2, 1]$.

It should of course be borne in mind that the regions of stability are only with respect to the simple test equations (3.6). Different behavior may occur for more complicated equations, particularly nonlinear equations. As in the deterministic context, some numerical experimentation may be required to find an appropriate scheme for a particular equation.

7. IMPLICIT TWO-STEP STRONG SCHEMES

The method to be indicated in Section 8 for deriving the above explicit and implicit schemes can also be used to construct implicit two-step schemes. We know from deterministic numerical analysis that the use of multistep methods allows a significant reduction in the computer time required for practical computations. The stochastic implicit two-step schemes which we shall describe here may be effective for stiff stochastic differential equations and offer similar advantages.

For the 1-dimensional case we have the following *Ito version* of an *implicit two-step order*-1.0 *strong scheme*:

$$Y_{n+1} = Y_{n-1} + [a(Y_{n+1}) + a]\Delta + V_n + V_{n-1} \tag{7.1}$$

with

$$V_n = b\,\Delta W_n + \tfrac{1}{2} bb'[(\Delta W_n)^2 - \Delta]$$

The *Stratonovich version* of this scheme is

$$Y_{n+1} = Y_{n-1} + [\underline{a}(Y_{n+1}) + \underline{a}]\Delta + V_n + V_{n-1} \tag{7.2}$$

with

$$\underline{V}_n = b\,\Delta W_n + \tfrac{1}{2} bb'(\Delta W_n)^2$$

and a *derivative-free Stratonovich version* is

$$Y_{n+1} = Y_{n-1} + [\underline{a}(Y_{n+1}) + \underline{a}]\Delta + \underline{V}_n + \underline{V}_{n-1} \tag{7.3}$$

with

$$\underline{V}_n = b\,\Delta W_n + \frac{1}{2\sqrt{\Delta}}[b(\bar{Y}_n) - b](\Delta W_n)^2$$

where

$$\bar{Y}_n = Y_n + \underline{a}\Delta + b\sqrt{\Delta}$$

There is a *family of implicit order*-1.0 *two step strong schemes* for which the *Ito version* in the *d*-dimensional case is given by

$$\begin{aligned} Y_{n+1} = {} & (1-\gamma)Y_n + \gamma Y_{n-1} \\ & + \{\alpha_2 a(Y_{n+1}) + [\gamma\alpha_1 + (1-\alpha_2)]a + \gamma(1-\alpha_1)\,a(Y_{n-1})\}\Delta \\ & + V_n + \gamma V_{n-1} \end{aligned} \tag{7.4}$$

with

$$V_n = b\,\Delta W_n + \tfrac{1}{2} L^1 b[(\Delta W_n)^2 - \Delta]$$

and parameters α_1, α_2, $\gamma \in [0, 1]$.

The *Stratonovich version* of (7.4) is

$$Y_{n+1} = (1-\gamma) Y_n + \gamma Y_{n-1} + \{\alpha_2 \underline{a}(Y_{n+1}) + [\gamma\alpha_1 + (1-\alpha_2)]\underline{a} + \gamma(1-\alpha_1)\underline{a}(Y_{n-1})\}\Delta + \underline{V}_n + \gamma \underline{V}_{n-1} \tag{7.5}$$

with

$$\underline{V}_n = b\,\Delta W_n + \tfrac{1}{2} L^1 b (\Delta W_n)^2 \tag{7.6}$$

and parameters $\alpha_1, \alpha_2, \gamma \in [0, 1]$. We obtain a *derivative-free version* of this scheme if instead of (7.6) we use

$$\underline{V}_n = b\,\Delta W_n + \frac{1}{2\sqrt{\Delta}} [b(\bar{Y}_n) - b](\Delta W_n)^2 \tag{7.7}$$

where

$$\bar{Y}_n = Y_n + \underline{a}\Delta + b\sqrt{\Delta}$$

We can control the stability of these schemes with an appropriate choice of the parameters α_1, α_2, and γ. The schemes (7.1)–(7.3) are obviously a special case with $\alpha_1 = 0$, $\alpha_2 = 1/2$, and $\gamma = 1$. Different values of the parameters can be used in the different components of the scheme if this is advantageous.

In the d-dimensional case an *implicit two-step order*-1.5 *strong scheme* is given by

$$Y_{n+1} = Y_{n-1} + \tfrac{1}{2}[a(Y_{n+1}) + 2a + a(Y_{n-1})]\Delta + V_n + V_{n-1} \tag{7.8}$$

with

$$V_n = b\,\Delta W_n + ab'(\Delta W_n \Delta - \Delta Z_n) + a'b(\Delta Z_n - \tfrac{1}{2}\Delta W_n \Delta) + \tfrac{1}{2} bb'[(\Delta W_n)^2 - \Delta] + \tfrac{1}{2} b(bb')'[\tfrac{1}{3}(\Delta W_n)^2 - \Delta]\,\Delta W_n$$

We have in the d-diensional case the following *family of implicit two-step order*-1.5 *strong schemes*

$$Y_{n+1} = (1-\gamma) Y_n + \gamma Y_{n-1} + \tfrac{1}{2}[a(Y_{n+1}) + (1+\gamma)a + \gamma a(Y_{n-1})]\Delta - \tfrac{1}{2}(1-\gamma) L^1 a(Y_{n-1})\,\Delta W_{n-1}\,\Delta + V_n + \gamma V_{n-1} \tag{7.9}$$

with

$$V_n = b\,\Delta W_n + L^0 b(\Delta W_n \Delta - \Delta Z_n) + L^1 a(\Delta Z_n - \tfrac{1}{2}\Delta W_n \Delta)$$
$$+ \tfrac{1}{2}L^1 b[(\Delta W_n)^2 - \Delta] + \tfrac{1}{2}L^1 L^1 b[\tfrac{1}{3}(\Delta W_n)^2 - \Delta]\,\Delta W_n$$

with parameter $\gamma \in [0, 1]$.

A *derivative-free version* of the above family of implicit two-step order-1.5 strong schemes (7.9) with $\gamma = 1$ is

$$Y_{n+1} = Y_{n-1} + \tfrac{1}{2}[a(Y_{n+1}) + 2a + a(Y_{n-1})]\Delta + V_n + V_{n-1} \quad (7.10)$$

with

$$V_n = b\,\Delta W_n + \frac{1}{2\sqrt{\Delta}}[a(\bar{Y}_n^+) - a(\bar{Y}_n^-)]\left(\Delta Z_n - \frac{1}{2}\Delta W_n \Delta\right)$$
$$+ \frac{1}{4\sqrt{\Delta}}[b(\bar{Y}_n^+) - b(\bar{Y}_n^-)][\Delta W_n)^2 - \Delta]$$
$$+ \frac{1}{2\Delta}[b(\bar{Y}_n^+) - 2b + b(\bar{Y}_n^-)](\Delta W_n \Delta - \Delta Z_n)$$
$$+ \frac{1}{4\Delta}[b(\bar{\Phi}_n^+) - b(\bar{\Phi}_n^-) - [b(\bar{Y}_n^+) - b(\bar{Y}_n^-)]]$$
$$\times \left[\frac{1}{3}(\Delta W_n)^2 - \Delta\right]\Delta W_n$$

with supporting values

$$\bar{Y}_n^{\pm} = Y_n + a\Delta \pm b\sqrt{\Delta}$$

and

$$\bar{\Phi}_n^{\pm} = \bar{Y}_n^+ \pm b(\bar{Y}_n^+)\sqrt{\Delta}$$

We shall see in the last section of the paper that it is also possible to derive analogous order-1.5 strong schemes with other degrees of implicitness than the $\gamma = 1$ used here.

As before, we shall consider only the Stratonovich versions of the order-2.0 schemes and restrict ourselves to equations with additive noise. For the 1-dimensional case we have the *implicit two-step order*-2.0 *strong scheme for additive noise*

$$Y_{n+1} = Y_{n-1} + \tfrac{1}{2}[\underline{a}(Y_{n+1}) + 2\underline{a} + \underline{a}(Y_{n-1})]\Delta + V_n + V_{n-1} \quad (7.11)$$

with

$$V_n = b\,\Delta W_n + \underline{a}'b(\Delta Z_n - \tfrac{1}{2}\Delta W_n\,\Delta) + \underline{a}''b^2[J_{(1,1,0),n} - \tfrac{1}{4}(\Delta W_n)^2\Delta]$$

We note that if $\underline{a}'' \equiv 0$, as in the linear case, then the final term in V_n involving the multiple integral $J_{(1,1,0),n}$ vanishes.

In the d-dimensional case there is a *family of implicit two-step order-*2.0 *strong schemes for additive noise* given by

$$\begin{aligned} Y_{n+1} = {} & (1-\gamma)\,Y_n + \gamma Y_{n-1} \\ & + \tfrac{1}{2}[\underline{a}(Y_{n+1}) + (1+\gamma)\underline{a} + \gamma\underline{a}(Y_{n-1})]\,\Delta \\ & - \tfrac{1}{2}(1-\gamma)\,L^1\underline{a}(Y_{n-1})\,\Delta W_{n-1}\,\Delta \\ & - \tfrac{1}{4}(1-\gamma)\,L^1L^1\underline{a}(\Delta W_n)^2\,\Delta \\ & + V_n + \gamma V_{n-1} \end{aligned} \tag{7.12}$$

with

$$V_n = b\,\Delta W_n + L^1\underline{a}(\Delta Z_n - \tfrac{1}{2}\Delta W_n\,\Delta) + L^1L^1\underline{a}[J_{(1,1,0),n} - \tfrac{1}{4}(\Delta W_n)^2\,\Delta]$$

where $\gamma \in [0, 1]$. This scheme simplifies considerably when $\gamma = 1$.

8. CONVERGENCE

The orders of strong convergence of the schemes presented in the preceding sections follow from a theorem in ref. 21, which is essentially a slight generalization of the main theorem in ref. 19. The same assertion and proof can also be found in ref. 12. It is based on a straightforward repeated application of the Ito–Taylor formula[22] or the Stratonovich–Taylor formula[11] combined with the mean square properties of multiple stochastic integrals. To be more precise, we formulate a strong convergence theorem for the order-2.0 strong Taylor scheme (4.13) which is a consequence of Corollary 10.7.2 in ref. 12.

Theorem. Let Y_n^{Δ} denote the value at time $\tau_n = n\Delta$ of the order-2.0 strong Taylor scheme (4.13) with step size Δ. Suppose that $\underline{a}$ and b are three and four times continuously differentiable, respectively, with all of these derivatives being uniformly bounded. In addition suppose that

$$E|X_0|^2 = \langle |X_0|^2 \rangle < \infty$$

and that

$$E|X_0 - Y_0^{\Delta}|^2 = \langle |X_0 - Y_0^{\Delta}|^2 \rangle \leqslant K_1 \Delta^4$$

for some constant K_1 which does not depend on Δ. Then there exists another constant $K_2 < \infty$ which does not depend on Δ such that

$$E \max_{n=0,\dots,n_T} |X_{\tau_n} - Y_n^{\Delta}| = \langle \max_{n=0,\dots,n_T} |X_{\tau_n} - Y_n^{\Delta}| \rangle \leqslant K_2 \Delta^2$$

where $n_T = T/\Delta$.

The conditions for the other schemes proposed above are quite similar. Their strong convergence is covered by corresponding theorems in ref. 12.

For the Stratonovich case we end up with the following d-dimensional family of implicit two-step expansions (8.1) which describes all of the terms needed in the derivation of an order-2.0 strong scheme. The other strong Taylor schemes can also be obtained from (8.1) by omitting those terms which are superfluous for the desired order of convergence. Proper approximation of the derivatives appearing in these Taylor schemes then yield the derivative-free schemes of the corresponding strong order.

For simplicity we shall use the abbreviations

$$X_n = X_{\tau_n}^{\tau_{n-1}, X_{n-1}}, \qquad X_{n+1} = X_{\tau_{n+1}}^{\tau_{n-1}, X_{n-1}}$$

and $f = f(X_n)$ for the functions $f = \underline{a}, b, \underline{L}^0\underline{a}, \dots$, where X_{n-1} is the value of X at τ_{n-1}. Then the announced expansion has the form

$$\begin{aligned} X_{n+1} = {} & (1-\gamma) X_n + \gamma X_{n-1} \\ & + [\alpha_2 \underline{a}(X_{n+1}) + (1 - \alpha_2 + \gamma\alpha_1)\underline{a} + \gamma(1-\alpha_1)\, \underline{a}(X_{n-1})]\Delta \\ & + \{(\tfrac{1}{2} - \alpha_2)\, \beta_2 \underline{L}^0 \underline{a}(X_{n+1}) \\ & + [(\tfrac{1}{2} - \alpha_2)(1-\beta_2) + \gamma(\tfrac{1}{2} - \alpha_1)\beta_1] \underline{L}^0 \underline{a} \\ & + \gamma(\tfrac{1}{2} - \alpha_1)(1-\beta_1)\, \underline{L}^0 \underline{a}(X_{n-1})\} \Delta^2 \\ & - \alpha_2 L^1 \underline{a}\, \Delta W_n \Delta - \gamma\alpha_1 L^1 \underline{a}(X_{n-1})\, \Delta W_{n-1} \Delta \\ & - \tfrac{1}{2}\gamma\alpha_2 L^1 L^1 \underline{a} (\Delta W_n)^2 \Delta \\ & - \tfrac{1}{2}\gamma\alpha_1 L^1 L^1 \underline{a}(X_{n-1})(\Delta W_{n-1})^2 \Delta \\ & + V_n + \gamma V_{n-1} \end{aligned} \tag{8.1}$$

with

$$V_n = b\,\Delta W_n + \underline{L}^0 b\{\Delta W_n \Delta - \Delta Z_n\} + \frac{1}{2} L^1 \underline{a}\,\Delta Z_n + \frac{1}{2} L^1 b(\Delta W_n)^2$$

$$+ \underline{L}^0 L^1 b J_{(0,1,1),n} + L^1 \underline{L}^0 b J_{(1,0,1),n} + L^1 L^1 \underline{a} J_{(1,1,0),n}$$

$$+ \frac{1}{3!} L^1 L^1 b(\Delta W_n)^3 + \frac{1}{4!} L^1 L^1 L^1 b(\Delta W_n)^4$$

$$+ \text{higher-order terms}$$

Here the parameters α_1, α_2, β_1, β_2, and γ can be chosen to be any finite real numbers, but we have only considered interpolation schemes in this paper, so these parameters were restricted to have values in the interval [0, 1]. Furthermore, from the derivation of (8.1) it is not difficult to see that these parameters may differ in the different components of the expansion. Assuming sufficient smoothness, linear growth, or boundedness conditions on the drift and diffusion coefficients it is straightforward to apply the strong convergence theorems mentioned at the beginning of this section. The desired order of strong convergence of a scheme then follows from the moment properties of the multiple stochastic integrals appearing in the scheme.[22] This was done in full detail for the strong Taylor approximations in ref. 19.

9. NUMERICAL RESULTS

Extensive and systematic numerical testing of the implicit schemes introduced in this paper, which will be reported elsewhere, confirm their better stability and convergence properties. Here we shall present some numerical results to provide the reader with an indication of what may be anticipated. For this we consider the 2-dimensional linear Ito differential equation with scalar noise

$$dX_t = AX_t\,dt + BX_t\,dW_t \tag{9.1}$$

where

$$A = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \quad \text{and} \quad B = 0.01I = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Since the matrices A and B commute, the solution[1,12] is given by

$$\begin{aligned} X_t &= X_0 \exp[(A - \tfrac{1}{2}B^2)t - BW_t] \\ &= X_0 \exp\left(\begin{bmatrix} -5.00005t + 0.01W_t & 4.99995t \\ 4.99995t & -5.00005t + 0.01W_t \end{bmatrix}\right) \end{aligned} \tag{9.2}$$

The Lyapunov exponents turn out to be the real parts of the eigenvalues of the matrix $A - \frac{1}{2}B^2$, that is,

$$\lambda_2 = -10.00005 \ll \lambda_1 = -0.00005$$

so (9.1) is quite stiff.

With the initial value $X_0 = (1, 0)$, (9.2) reduces to

$$X_t = (\tfrac{1}{2}(e^{\lambda^+(t)} + e^{\lambda^-(t)}), \tfrac{1}{2}(e^{\lambda^+(t)} - e^{\lambda^-(t)})) \tag{9.3}$$

where

$$\lambda^+(t) = -0.00005t + 0.01 W_t, \qquad \lambda^-(t) = -10.00005t + 0.01 W_t$$

We applied the numerical schemes to (9.1) with the above initial condition for equal time steps Δ over the interval [0, 1] and evaluated the absolute error at the end of the interval. We did this for $M = 200$ batches of $N = 10$ trajectories for each scheme and choice of time step. Let

$$E_{i,j} = |X_T(\omega_{i,j}) - Y^{\Delta}(\omega_{i,j})|$$

be the absolute error for the jth trajectory of the ith batch and let

$$E_i = \frac{1}{N} \sum_{j=1}^{N} E_{i,j}, \qquad E = \frac{1}{M} \sum_{i=1}^{M} E_i$$

be the sample means of the ith batch and of all batches, respectively. From the Student's t-distribution with $M-1$ degrees of freedom, the 90% confidence interval $(E(-), E(+))$ for the sample mean has endpoints given by

$$E(\pm) = E \pm t_{0.1, M-1} \left(\frac{S^2}{10}\right)^{1/2}$$

where $t_{0.1, 199} = 1.65$ and the sample variance is

$$S^2 = \frac{1}{M-1} \sum_{i=1}^{M} (E_i - E)^2$$

The calculations were repeated for time steps $\Delta = 2^{-2}$, 2^{-3}, 2^{-4}, and 2^{-5}. The confidence intervals obtained for the simulations for the different schemes turned out to be quite small, that is, with $E(+) - E(-) < 10^{-4}$. We plotted $\log_2 |E|$ against $\log_2 \Delta$ and present nine figures here which are representative of the results. In each case the order of convergence can be easily seen from the slope of the curve in the figure. We also include tables of the time steps, computed absolute values E, and error bars for each scheme considered (Tables I–VII).

Table I. Implicit Euler Scheme with $\alpha = 0$

Δ	Mean error E	Error bar
0.25	3.579860498	±0.000849423
0.125	0.000026558	±0.000000233
0.0625	0.000034093	±0.000000119
0.03125	0.000029013	±0.000000078

Table II. Implicit Euler Scheme with $\alpha = 0.5$

Δ	Mean error E	Error bar
0.25	0.000079582	±0.000000304
0.125	0.000031197	±0.000000323
0.0625	0.000014515	±0.000000117
0.03125	0.000007803	±0.000000174

Table III. Implicit Milstein Scheme with $\alpha = 0$

Δ	Mean error E	Error bar
0.25	3.579909532	±0.0009929449
0.125	0.000021314	±0.000000028
0.0625	0.000032007	±0.000000011
0.03125	0.000027718	±0.000000008

Table IV. Implicit Milstein Scheme with $\alpha = 0.5$

Δ	Mean error E	Error bar
0.25	0.000075755	±0.000000164
0.125	0.000026418	±0.000000005
0.0625	0.00000939	±0.000000001
0.03125	0.000002545	±0.000000001

Table V. Implicit Order-1.5 Strong Taylor Scheme with $\alpha = 0.5$

Δ	Mean error E	Error bar
0.25	0.000075642	$\pm$0.000000027
0.125	0.000026404	$\pm$0.00000001
0.0625	0.000009388	$\pm$0.000000003
0.03125	0.000002545	$\pm$0.000000001

Table VI. Implicit Two-Step Order-1.0 Scheme with $\alpha = \gamma = 1.0$

Δ	Mean error E	Error bar
0.25	0.004647822	$\pm$0.00005869
0.125	0.001145033	$\pm$0.00002175
0.0625	0.000427624	$\pm$0.000013339
0.03125	0.000181653	$\pm$0.000004805

Table VII. Implicit Two-Step Order-1.0 Scheme with $\alpha = 0.5$

Δ	Mean error E	Error bar
0.25	0.003515983	$\pm$0.00010349
0.125	0.001227402	$\pm$0.00003581
0.0625	0.000436288	$\pm$0.000012377
0.03125	0.0001512	$\pm$0.000004131

The results for the implicit Euler scheme with implicitness parameters $\alpha_1 = \alpha_2 = 0.0$, 0.5, and 1.0 are presented in Figs. 1–3, respectively. We note that the explicit Euler scheme ($\alpha_i = 0.0$) does not work for step sizes $\Delta \geqslant 2^{-3}$. In contrast, the implicit Eulei schemes with $\alpha_1 = \alpha_2 = 0.5$ and 1.0 converge for all of the considered step sizes with order 0.5 or even better.

Analogous results for the implicit Milstein scheme with parameters $\alpha_1 = \alpha_2 = 0.0$, 0.5, and 1.0 are presented in Figs. 4–6, respectively, except here we observe the strong order of convergence 1.0 for the implicit

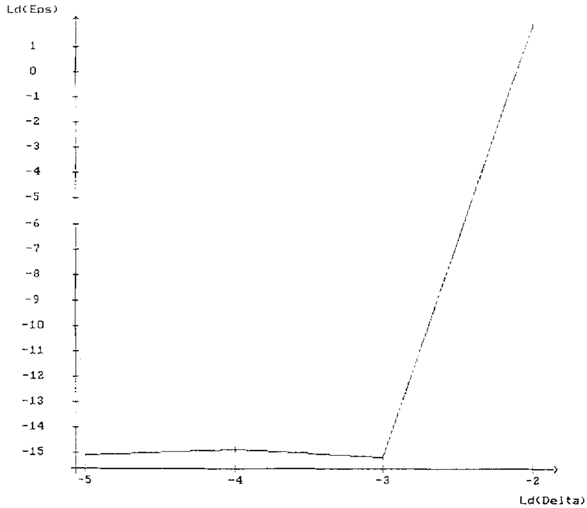

Fig. 1. Implicit Euler scheme with $\alpha = 0$.

versions with $\alpha_i = 0.5$ and 1.0. The order-1.0 Runge-Kutta scheme gave similar results to those of the Milstein scheme, so they have not been included here.

More interesting is Fig. 7 for the order-1.5 implicit strong Taylor scheme with $\alpha = 0.5$, which shows an order of strong convergence at least

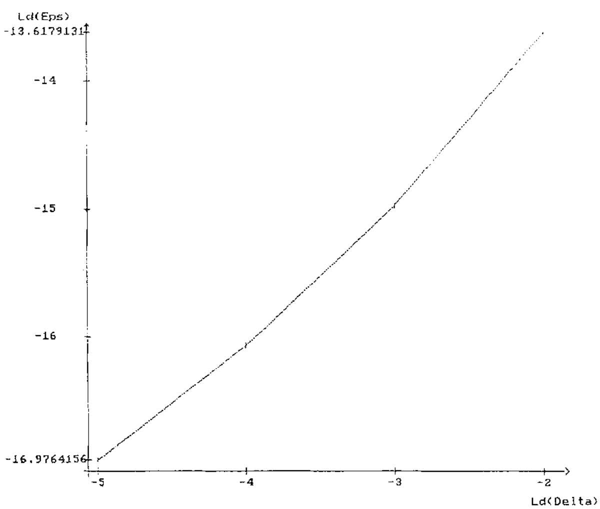

Fig. 2. Implicit Euler scheme with $\alpha = 0.5$.

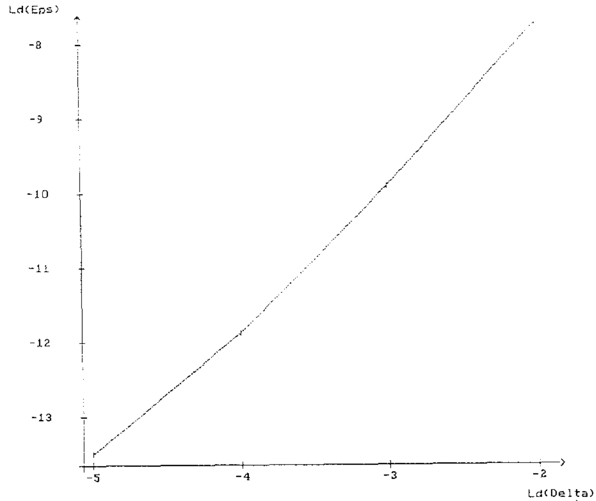

Fig. 3. Implicit Euler scheme with $\alpha = 1.0$.

as large as 1.5. Similar results were obtained for the order-1.5 strong implicit Taylor scheme with $\alpha = 0.5$.

Figure 8 contains the results for the order-1.0 implicit two-step scheme with parameters $\alpha = \gamma = 1.0$, where the Milstein scheme was used as the starting routine. Finally, Fig. 9 gives the result for the order-1.5 implicit

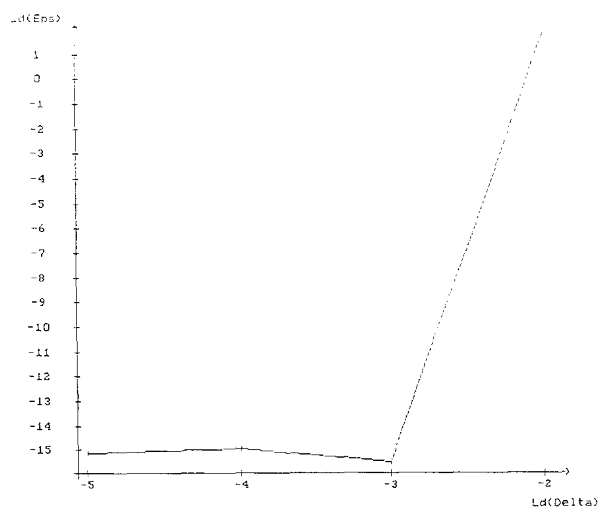

Fig. 4. Implicit Milstein scheme with $\alpha = 0$.

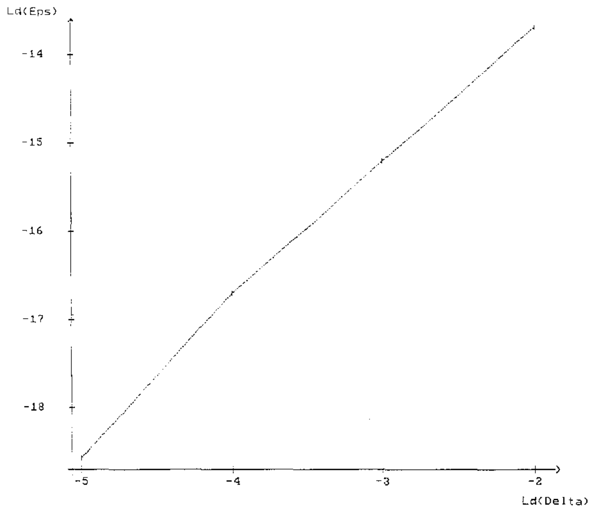

Fig. 5. Implicit Milstein scheme with $\alpha = 0.5$.

two-step scheme with $\gamma = 1.0$, where the order-1.5 implicit strong Taylor scheme was used as the starting routine.

Comparing the simulations for this example, we see that best results are obtained with the implicit Milstein scheme with $\alpha_i = 0.5$, the order-1.5 implicit strong Taylor scheme with $\alpha_i = 0.5$, and the order-1.0 implicit two-step scheme with $\gamma = 1.0$.

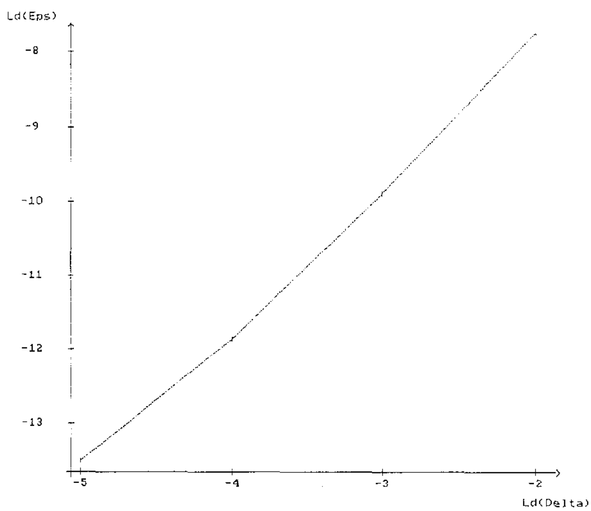

Fig. 6. Implicit Milstein scheme with $\alpha = 1.0$.

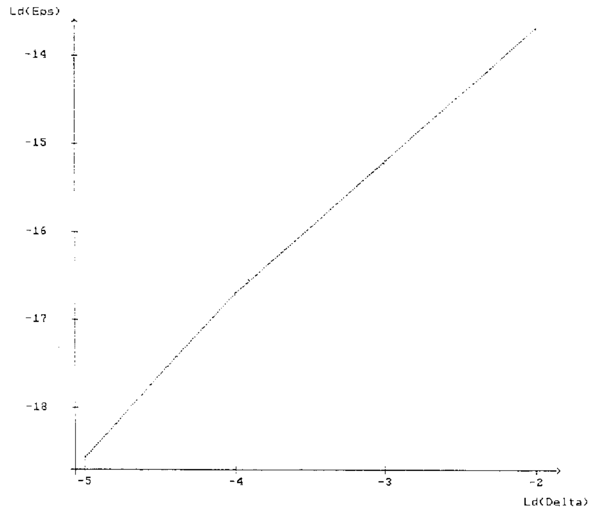

Fig. 7. Implicit order-1.5 strong Taylor scheme with $\alpha = 0.5$.

The results provide a clear indication of the order of convergence of the schemes and of the improvement in stability obtained by using an implicit scheme. Note that Euler, Milstein, and order-1.0 Runge-Kutta schemes with $\alpha_i = 0$ are in fact fully explicit schemes and are highly unstable for insufficiently small step sizes.

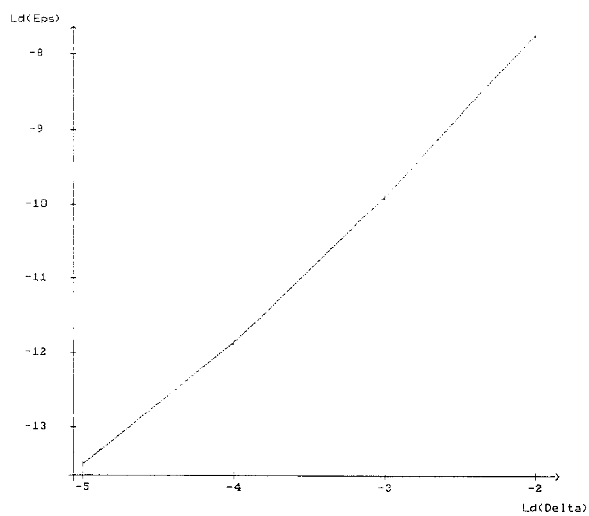

Fig. 8. Implicit order-1.0 two-step scheme with $\alpha = \gamma = 1.0$.

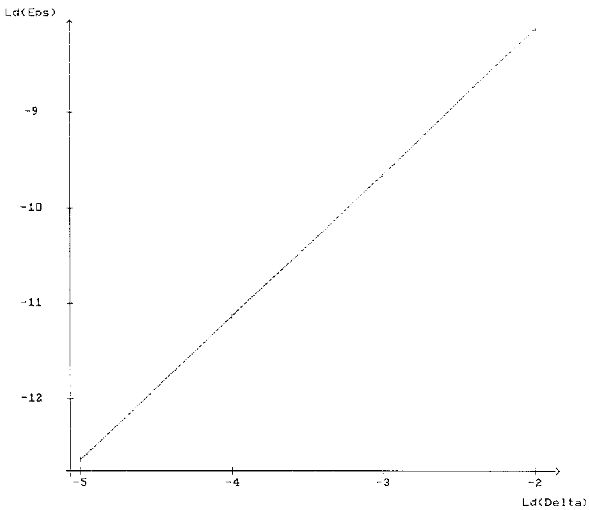

Fig. 9. Implicit order-1.0 two-step scheme with $\alpha = 0.5$.

10. CONCLUDING REMARKS

All of the schemes that we have introduced in this paper follow from the general expansions formula (8.1). Other strong schemes, including derivative-free ones, can also be derived from (8.1). This, however, is more conveniently done for special classes of stochastic differential equations which allow simplifications in the formulation of the schemes. In principle the schemes presented above can be generalized to the case of a multidimensional Wiener process driving the stochastic differential equation. However, the schemes then include multiple stochastic integrals which can be approximated by, for example, the method proposed in Kloeden *et al.*[13]

A fundamental difference in the analysis of deterministic and stochastic numerical schemes is that the latter need not only have sufficient smoothness of the coefficients, but must also include higher-order multiple stochastic integrals in order to achieve a higher order of convergence. The inclusion of multiple stochastic integrals in higher-order stochastic numerical schemes is dictated by the necessity of approximating the Wiener chaos to the desired order. Heuristic adaptations of deterministic numerical methods, such as the deterministic Runge-Kutta methods, usually have not included such multiple stochastic integrals and so cannot in general achieve a higher order of strong convergence than the stochastic Euler scheme. The stochastic Taylor formula provides the appropriate tool for the derivation of higher-order stochastic numerical schemes.

The implicit schemes that have been used by other authors, such as Klauder and Petersen,[8] Petersen,[18] Drummond and Mortimer,[5] Smith and Gardiner,[24] and McNeil and Craig,[14] in simulation studies are all included in our classes of schemes, being in fact simpler lower-order representatives. These authors investigated the stability and efficiency of their schemes for particular test equations of interest to them. The additional schemes proposed in this paper are higher-order one-step or two-step schemes, and so offer potentially greater efficiency and stability. A detailed stability analysis of the proposed two-step schemes and multistep schemes in general was beyond the scope of this paper.

ACKNOWLEDGMENTS

The authors thank H. Schurz for the computation of the numerical examples. We also thank the referee for several useful suggestions.

REFERENCES

1. L. Arnold, *Stochastic Differential Equations* (Wiley, New York, 1974).
2. L. Arnold and V. Wihstutz, eds., *Lyapunov Exponents* (Springer Lecture Notes in Mathematics, Vol. 1186, 1986).
3. Chien-Cheng Chang, Numerical solution of stochastic differential equations with constant diffusion coefficients, *Math. Comput.* **49**:523–542 (1987).
4. G. Dahlquist, A special stability problem for linear multistep methods, *BIT* **3**:27–43 (1963).
5. P. D. Drummond and I. K. Mortimer, Computer simulation of multiplicative stochastic differential equations, *J. Comput. Phys.*, to appear.
6. A. Greiner, W. Strittmatter, and J. Honerkamp, Numerical integration of stochastic differential equations, *J. Stat. Phys.* **51**:95–108 (1987).
7. R. Z. Hasminski, *Stochastic Stability of Differential Equations* (Sijthoff & Noordhoff, Alphen aan den Rijn, 1980).
8. D. B. Hernandez and R. Spigler, Numerical stability of implicit Runge-Kutta methods for stochastic differential equations, Preprint, University of Padua (1989).
9. J. R. Klauder and W. P. Petersen, Numerical integration of multiplicative-noise stochastic differential equations, *SIAM J. Numer. Anal.* **22**:1153–1166 (1985).
10. P. E. Kloeden and E. Platen, A survey of numerical methods for stochastic differential equations, *J. Stoch. Hydrol. Hydraul.* **3**:155–178 (1989).
11. P. E. Kloeden and E. Platen, The Stratonovich and Ito–Taylor expansions, *Math. Nachr.* **151**: 33–50 (1991).
12. P. E. Kloeden and E. Platen, *The Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Applications of Mathematics Series No. 23, 1991.
13. P. E. Kloeden, E. Platen, and I. Wright, The approximation of multiple stochastic integrals, *J. Stoch. Anal. Appl.*, to appear.
14. K. J. McNeil and I. J. D. Craig, An unconditional stable numerical scheme for nonlinear quantum optical systems—Quantised liit cycle behaviour in second harmonic generation, University of Waikato Research Report No. 166 (1988).

15. G. N. Milstein, Approximate integration of stochastic differential equations, *Theor. Prob. Appl.* **19**:557–562 (1974).
16. G. N. Milstein, *The Numerical Integration of Stochastic Differential Equations* (Urals University Press, Sverdlovsk, USSR, 1988) [in Russian].
17. E. Pardoux and D. Talay, Discretization and simulation of stochastic differential equations, *Acta Appl. Math.* **3**:23–47 (1985).
18. W. P. Petersen, Stability and accuracy of simulations for stochastic differential equations, IPS Research Report No. 90-02, ETH Zürich (1990).
19. E. Platen, A Taylor formula for semimartingales solving a stochastic differential equation, in *Springer Lecture Notes in Control and Information Science*, Vol. 36, pp. 163–172 (1981).
20. E. Platen, A generalized Taylor formula for solutions of stochastic differential equations, *Sankhya* **44A**:163–172 (1982).
21. E. Platen, Zur zeitdiskreten Approximation von Itoprozessen, Diss. B., IMath, Akademie der Wissenschaften der DDR, Berlin (1984).
22. E. Platen and W. Wagner, On a Taylor formula for a class of Ito processes, *Prob. Math. Stat.* **3**:37–51 (1982).
23. W. Rümelin, Numerical treatment of stochastic differential equations, *SIAM J. Numer. Anal.* **19**:604–613 (1982).
24. A. M. Smith and C. W. Gardiner, Simulation of nonlinear quantum damping using the positive representation, University of Waikato Research Report (1988).
25. D. Talay, Analyse Numérique des Equations Différentielles Stochastiques, Thèse 3ème cycle, Université Provence (1982).
26. W. Wagner and E. Platen, Approximation of Ito integral equations, Preprint ZIMM, Akedemie der Wissenschaften der DDR, Berlin (1978).